Set Theory

Set theory was developed by mathematicians to map degrees of infinity. Later it was used to model the whole of mathematics, and reveal much of its reasoning. Analytic metaphysicians then took an interest, because it might reveal an abstract foundation to reality, and it offered a rigorous way to model theories. In its early days sets were defined by concepts (such as everything that is 'red'), but this hit paradoxical problems, and so a more cautious system emerged, known as 'Zermelo-Fraenkel with Choice' (**ZFC**), which we will outline.

Set theory uses classical logic, with one further connective, '∈', read as 'is a member of'. Sets are denoted by curly brackets (e.g. **{**1,3,5,22**}**). Set theory is defined by nine axioms, saying what sets are, and what we can do with them:

I. Extensionality: *if two sets have the same members, they are the same set*. It is taken as basic that the members define the set. Members are objects, so each set has a 'cardinality' (a total of members). A problem is that we want to talk of the set of all insects, but don't know its cardinality.

II. Pairing: *for any two sets there is a third set they belong to*. This seems obvious, but hard to justify. Each member has a 'singleton' set {x}, which is defined by Pairing as {x,x}.

III. Unions: *in a collection of sets, all the elements can form a single set*. Also fairly obvious, but hard to justify.

IV. Empty or Null Set: *there exists a set ('∅') with no members*. This is useful in proofs, and is the first building block in the 'hierarchy of sets' (called '**V**'), but it is puzzling. If sets are collections, the null set doesn't collect anything.

V. Infinity: *when elements have 'successors', they are all in one set*. This allows the infinity of natural numbers to be treated as a single set in mathematics. Because it asserts the existence of something, it prevents set theory from counting as 'pure logic'.

VI. Powers: *in a collection of sets, all the subsets can form a single set*. Compare this with 'Union'. There are far more subsets than members, so the powers axiom generates huge 'power sets'. Since the axiom can always be applied again, this generates the endless and vast hierarchy (V) of sets. Is hard to object to the procedure, but the result is beyond human comprehension, so there are doubts about the axiom.

VII. Replacement: *any function applied to a set will produce another set*. Not controversial, but only needed for advanced mathematical proofs about large infinities.

VIII. Foundation: *descending chains of membership are finite in length*. This is the key axiom for the 'iterative conception' of sets, in which each level of the hierarchy is built up from the bottom, and is thus securely founded and free from paradox.

XI. Choice: *a set may be created by choosing one member from each subset*. This allows choice even when there is no principle for the choosing. This was controversial for a long time, because the freedom it allows is less obvious, but it was so useful in mathematics that it is now standard.

If A is a subset of B, we write $A \subseteq B$ (which also allows A and B to be the same), and $A \subset B$ means A is a 'proper subset' (not the same as B). Two sets can form a 'union', combining all the members, written as $A \cup B$. They can be 'disjoint' if they have no members in common. They can produce an 'intersection', of those members they share, written as $A \cap B$. A 'function' (F(x)) takes set members as input, and has a single output. It can be repeatedly applied, to produce a new set or change the members of the initial set. There is array of vocabulary to describe the way in which functions 'map' sets onto one another, which leads to Model Theory. A set is normally written by listing the members ('enumeration'), such as $\{a_1, a_2, a_3, \ldots\ldots a_n\}$, but it can also be written as fulfilling a condition ('abstraction'), such as {x: P(x)}, meaning 'the set of x's which have property P'.

A key principle is the distinction between the 'membership' and 'subset' relations. A set is a member of another set as a single object, but it is a subset of a set if its members are 'included' in that bigger set. Thus the U.S. is a member of the U.N., and a subset of North America (because its citizens are not U.N. members, but are included among North Americans). A set is 'transitive' if the members of members are included in the set. ZFC is said to be 'pure' set theory, because it forgets about basic elements, and just deals with the structures of the sets.

An ordered pair is written as <a,b>, meaning 'a then b'. It was discovered that {a,{a,b}} exactly copies the behaviour of <a,b>, which means that ordering can be handled in ZFC. A 'total' ordering of every element is fixed by the 'greater than' relation, and 'a' must always be less than, equal to, or greater than 'b' (i.e. it obeys 'trichotomy'). A 'well-ordering' is a total ordering, with every subset having a first member. Given the Axiom of Infinity, the natural numbers can form a well-order set, starting with 0 or 1, which is why set theory can model arithmetic.

Philosophers who aim for a minimum ontology have noticed that properties can be modelled in sets, by just saying that 'red' is the set of the red objects. In this way, one might have a physical ontology that consisted of nothing but objects and sets (with the whole of mathematics as a bonus). However, what exactly *is* a set? Apparently if my ontology contains three objects A, B and C, I find myself with seven uninvited sets – {A},{B},{C},{A,B},{B,C},{A,C},{A,B,C}. What is the difference between object A, and its singleton set {A}?

A big attraction of set theory for philosophers is that it may the basis of all of mathematics, which seems like a foundation of nature, but the wild open-endedness of the hierarchy V means that the world of sets is not defined, and lacks a single structure (because it is not fully 'categorical'). Hence debate continues about the best axioms for sets.

The earliest set theory was Naïve Set Theory, but that met paradoxes. Now we have the Iterative Conception, but nervousness about the upper reaches leads to the Limitation of Size proposal, or the development of alternative versions, such as 'NBG' or 'NF'. One modern version suggests dropping the Powers Axiom, and numerous rival axioms have been suggested, either to restrain the system, or to justify its expansion (with Large Cardinal Axioms). Set theory is a specialist area, but it contains an excitement that grips philosophers.